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NOTE

On Hilbert's Integral Inequality

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In this paper, we generalize Hilbert's integral inequality and its equivalent form by introducing three parameters t, a, and b.

If $f, g \in L^2[0, \infty)$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy \le \pi\left(\int_0^\infty f^2(x)\,dx \int_0^\infty g^2(x)\,dx\right)^{1/2}, \tag{1}$$

where π is the best value. The inequality (1) is well known as Hilbert's integral inequality, and its equivalent form is

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\,dx\right)^2 dy \le \pi^2 \int_0^\infty f^2(x)\,dx, \tag{2}$$

where π^2 is also the best value (cf. [1, Chap. 9]). Recently, Hu Ke made the following improvement of (1) by introducing a real function $c(x)$,

$$\begin{aligned}\left|\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy\right|^4 &\le \pi^4\left\{\left(\int_0^\infty f^2(x)\,dx\right)^2 - \left(\int_0^\infty f^2(x)k(x)\,dx\right)^2\right\}\\ &\quad\times\left\{\left(\int_0^\infty g^2(x)\,dx\right)^2 - \left(\int_0^\infty g^2(x)k(x)\,dx\right)^2\right\},\end{aligned}$$

where $k(x) = 2/\pi\int_0^\infty (c(t^2x)/(1+t^2))\,dt - c(x), 1 - c(x) + c(y) \ge 0$, and $f, g \ge 0$ (cf. [2]). In this paper, some generalizations of (1) and (2) are given in the following theorems, which are other than those in [2].

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THEOREM 1. *Let $b > a > 0$, and $0 < t \le 1$, $f, g \in L^2[0,\infty)$. Then*

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^t}\,dx\,dy \le k_t\left[1-\left(\frac{a}{b}\right)^{t/4}\right] \times\left(\int_a^b x^{1-t}f^2(x)\,dx\int_a^b x^{1-t}g^2(x)\,dx\right)^{1/2}; \tag{3}$$

$$\int_0^b\int_0^b \frac{f(x)g(y)}{(x+y)^t}\,dx\,dy \le k_t\left\{\int_0^b\left[1-\frac{1}{2}\left(\frac{x}{b}\right)^{t/2}\right]x^{1-t}f^2(x)\,dx \times\int_0^b\left[1-\frac{1}{2}\left(\frac{x}{b}\right)^{t/2}\right]x^{1-t}g^2(x)\,dx\right\}^{1/2}; \tag{4}$$

$$\int_a^\infty\int_a^\infty \frac{f(x)g(y)}{(x+y)^t}\,dx\,dy \le k_t\left\{\int_a^\infty\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{t/2}\right]x^{1-t}f^2(x)\,dx \times\int_a^\infty\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{t/2}\right]x^{1-t}g^2(x)\,dx\right\}^{1/2}, \tag{5}$$

where $k_t = \int_0^\infty (1/(1+u)^t)(1/u)^{1-t/2}\,du = B(t/2, t/2)$, $B(p,q)$ $(p,q>0)$ *is the* β*-function. Especially for* $t = 1$, *we have*

$$\int_a^b\int_a^b \frac{f(x)g(y)}{x+y}\,dx\,dy \le \pi\left(1-\sqrt[4]{\frac{a}{b}}\right)\left(\int_a^b f^2(x)\,dx\int_a^b g^2(x)\,dx\right)^{1/2}; \tag{6}$$

$$\int_0^b\int_0^b \frac{f(x)g(y)}{x+y}\,dx\,dy \le \pi\left[\int_0^b\left(1-\frac{1}{2}\sqrt{\frac{x}{b}}\right)f^2(x)\,dx \times\int_0^b\left(1-\frac{1}{2}\sqrt{\frac{x}{b}}\right)g^2(x)\,dx\right]^{1/2}; \tag{7}$$

$$\int_a^\infty\int_a^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy \le \pi\left[\int_a^\infty\left(1-\frac{1}{2}\sqrt{\frac{a}{x}}\right)f^2(x)\,dx \times\int_a^\infty\left(1-\frac{1}{2}\sqrt{\frac{a}{x}}\right)g^2(x)\,dx\right]^{1/2}. \tag{8}$$

THEOREM 2. *Let $b > a > 0$, and $f \in L^2[0,\infty)$. Then*

$$\int_a^b \left(\int_a^b \frac{f(x)}{x+y}\, dx \right)^2 dy \le \pi^2 \left(1 - \sqrt[4]{\frac{a}{b}} \right)^2 \int_a^b f^2(x)\, dx; \tag{9}$$

$$\int_0^b \left(\int_0^b \frac{f(x)}{x+y}\, dx \right)^2 dy \le \pi^2 \int_0^b \left(1 - \frac{1}{2}\sqrt{\frac{x}{b}} \right) f^2(x)\, dx; \tag{10}$$

$$\int_a^\infty \left(\int_a^\infty \frac{f(x)}{x+y}\, dx \right)^2 dy \le \pi^2 \int_a^\infty \left(1 - \frac{1}{2}\sqrt{\frac{a}{x}} \right) f^2(x)\, dx. \tag{11}$$

Remark 1. When $a \to 0, b \to \infty$, (3) changes to

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^t}\, dx\, dy$$
$$\le k_t \left(\int_0^\infty x^{1-t} f^2(x)\, dx \int_0^\infty x^{1-t} g^2(x)\, dx \right)^{1/2} \qquad (0 < t \le 1). \tag{12}$$

When $t = 1$, (12) changes to (1). It is obvious that (12) is a generalization of (1), so are (3)–(8); and (9)–(11) are generalizations of (2).

In showing these, we consider some lemmas.

LEMMA 1. *For parameter t $(0 < t \le 1)$, define k_t and θ_t as*

$$k_t := \int_0^\infty \frac{1}{(1+u)^t} \left(\frac{1}{u} \right)^{1-t/2} du; \qquad \theta_t := \int_0^1 \frac{1}{(1+u)^t} \left(\frac{1}{u} \right)^{1-t/2} du. \tag{13}$$

Then $k_t = B(t/2, t/2)$, and $\theta_t = k_t/2$, where $B(p,q)$ $(p, q > 0)$ is the β-function.

Proof. Since the β-function

$$B(p,q) := \int_0^1 x^{p-1}(1-x)^{q-1}\, dx = \int_0^1 \frac{y^{p-1} + y^{q-1}}{(1+y)^{p+q}}\, dy \qquad (p, q > 0),$$

then $\theta_t = B(t/2, t/2)/2$. Putting $y = 1/u$, we find that

$$k_t = \theta_t + \int_1^\infty \frac{1}{(1+u)^t} \left(\frac{1}{u} \right)^{1-t/2} du$$
$$= \theta_t + \int_0^1 \frac{y^{t/2-1}}{(1+y)^t}\, dy = 2\theta_t = B\left(\frac{t}{2}, \frac{t}{2} \right).$$

The lemma is proved.

LEMMA 2. *For parameter* t $(0 < t \le 1)$, *define* $h(y)$ *as*

$$h(y) = y^{-t/2} \int_0^y \frac{1}{(1+u)^t} \left(\frac{1}{u}\right)^{1-t/2} du, \qquad \text{for } y \in (0,1].$$

Then

$$h(y) \ge h(1) = \theta_t \qquad (y \in (0,1]). \tag{14}$$

The equality contained in (14) *holds only when* $y = 1$.

Proof. For $y \in (0,1]$, we have

$$\begin{aligned} h'(y) &= -\frac{t}{2} y^{-1-t/2} \int_0^y \frac{1}{(1+u)^t} \left(\frac{1}{u}\right)^{1-t/2} du \\ &\quad + y^{-t/2} \frac{1}{(1+y)^t} \left(\frac{1}{y}\right)^{1-t/2} \\ &= -y^{-1-t/2} \int_0^y \frac{1}{(1+u)^t} du^{t/2} + y^{-1} \frac{1}{(1+y)^t} \\ &= -y^{-1-t/2} \left[\frac{1}{(1+u)^t} u^{t/2} \right] \Bigg|_0^y + y^{-1-t/2} \int_0^y u^{t/2} d \frac{1}{(1+u)^t} \\ &\quad + y^{-1} \frac{1}{(1+y)^t} \\ &= -y^{-1-t/2} \frac{1}{(1+y)^t} y^{t/2} - t y^{-1-t/2} \int_0^y u^{t/2} \frac{1}{(1+u)^{t+1}} du \\ &\quad + y^{-1} \frac{1}{(1+y)^t} \\ &= -t y^{-1-t/2} \int_0^y u^{t/2} \frac{1}{(1+u)^{t+1}} du < 0. \end{aligned}$$

Then $h(y)$ is strictly decreasing on $(0,1]$. Hence $h(y) \ge h(1) = \theta_t$, and the equality holds only when $y = 1$. This proves the lemma.

LEMMA 3. *Let* $b > a > 0$, *and* $0 < t \le 1$. *Define the weight function* $\omega_t(a,b,x)$ *as*

$$\omega_t(a,b,x) = \int_a^b \frac{1}{(x+y)^t} \left(\frac{x}{y}\right)^{1-t/2} dy, \qquad \text{for } x \in [a,b].$$

Then

$$\omega_t(a,b,x) < k_t\left[1-\left(\frac{a}{b}\right)^{t/4}\right]x^{1-t} \qquad (x\in[a,b]); \tag{15}$$

$$\omega_t(0,b,x) = \lim_{a\to 0}\omega_t(a,b,x) \le k_t\left[1-\frac{1}{2}\left(\frac{x}{b}\right)^{t/2}\right]x^{1-t} \qquad (x\in[0,b]); \tag{16}$$

$$\omega_t(a,\infty,x) = \lim_{b\to\infty}\omega_t(a,b,x) \le k_t\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{t/2}\right]x^{1-t} \qquad (x\in[a,\infty)), \tag{17}$$

where the constant k_t *is indicated as in Lemma* 1. *Especially for* $t = 1$, *we have*

$$\omega_1(a,b,x) < \pi\left(1-\sqrt[4]{\frac{a}{b}}\right) \qquad (x\in[a,b]); \tag{18}$$

$$\omega_1(0,b,x) \le \pi\left(1-\frac{1}{2}\sqrt{\frac{x}{b}}\right) \qquad (x\in(0,b]); \tag{19}$$

$$\omega_1(a,\infty,x) \le \pi\left(1-\frac{1}{2}\sqrt{\frac{a}{x}}\right) \qquad (x\in[a,\infty)); \tag{20}$$

Proof. Putting $u = y/x$, then we have

$$\begin{aligned}
\omega_t(a,b,x) &= \int_a^b \frac{1}{(x+y)^t}\left(\frac{x}{y}\right)^{1-t/2} dy \\
&= x^{1-t}\int_{a/x}^{b/x}\frac{1}{(1+u)^t}\left(\frac{1}{u}\right)^{1-t/2} du \\
&= x^{1-t}\left[k_t - \int_0^{a/x}\frac{1}{(1+u)^t}\left(\frac{1}{u}\right)^{1-t/2} du \right.\\
&\qquad \left. - \int_{b/x}^{\infty}\frac{1}{(1+u)^t}\left(\frac{1}{u}\right)^{1-t/2} du\right]
\end{aligned}$$

$$= x^{1-t}\left\{k_t - \left[\int_0^{a/x} \frac{1}{(1+u)^t}\left(\frac{1}{u}\right)^{1-t/2} du + \int_0^{x/b} \frac{1}{(1+\nu)^t}\left(\frac{1}{\nu}\right)^{1-t/2} d\nu\right]\right\}$$

$$= x^{1-t}\left\{k_t - \left[\left(\frac{a}{x}\right)^{t/2} h\left(\frac{a}{x}\right) + \left(\frac{x}{b}\right)^{t/2} h\left(\frac{x}{b}\right)\right]\right\}. \quad (21)$$

By Lemmas 1, 2, and the arithmetic-geometric average inequality, we have

$$\begin{aligned}\omega_t(a,b,x) &< x^{1-t}\left\{k_t - \left[\left(\frac{a}{x}\right)^{t/2} h(1) + \left(\frac{x}{b}\right)^{t/2} h(1)\right]\right\} \\ &= x^{1-t}\left\{k_t - \theta_t\left[\left(\frac{a}{x}\right)^{t/2} + \left(\frac{x}{b}\right)^{t/2}\right]\right\} \\ &\le x^{1-t}\left\{k_t - 2\theta_t\left[\left(\frac{a}{x}\right)^{t/2}\left(\frac{x}{b}\right)^{t/2}\right]^{1/2}\right\} \\ &= k_t\left[1 - \left(\frac{a}{b}\right)^{t/4}\right]x^{1-t} \qquad (x \in [a,b]).\end{aligned}$$

Relation (15) is valid. When $a \to 0$, by (21) and Lemmas 1, 2, we have

$$\begin{aligned}\omega_t(0,b,x) &\le x^{1-t}\left[k_t - \left(\frac{x}{b}\right)^{t/2} h\left(\frac{x}{b}\right)\right] \\ &\le x^{1-t}\left[k_t - \left(\frac{x}{b}\right)^{t/2} h(1)\right] \\ &\le k_t\left[1 - \frac{1}{2}\left(\frac{x}{b}\right)^{t/2}\right]x^{1-t} \qquad (x \in [0,b]).\end{aligned}$$

Relation (16) is valid and (17) holds by the same method. Since $k_1 = \pi$, and $\theta_1 = \pi/2$, substituting $t = 1$ in (15), (16), and (17), we have (18), (19), and (20). This proves the lemma.

Proof of Theorem 1. We use Hilbert's technique and apply Cauchy's inequality to estimate the left-hand side of (3) as

$$\begin{aligned}
&\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^t}\,dx\,dy \\
&\quad= \int_a^b\int_a^b \frac{f(x)}{(x+y)^{t/2}}\left(\frac{x}{y}\right)^{(1-t/2)/2} \frac{g(y)}{(x+y)^{t/2}}\left(\frac{y}{x}\right)^{(1-t/2)/2} dx\,dy \\
&\quad\le \left[\int_a^b\int_a^b \frac{f^2(x)}{(x+y)^t}\left(\frac{x}{y}\right)^{1-t/2} dx\,dy \int_a^b\int_a^b \frac{g^2(y)}{(x+y)^t}\left(\frac{y}{x}\right)^{1-t/2} dx\,dy\right]^{1/2} \\
&\quad= \left[\int_a^b \omega_t(a,b,x)f^2(x)\,dx \int_a^b \omega_t(a,b,y)g^2(y)\,dy\right]^{1/2}. \qquad (22)
\end{aligned}$$

By (15), the inequality (3) is valid. When $a \to 0$, by (22) and (16), (4) is valid. Relation (5) holds by the same method. Substituting $t = 1$ in (3), (4), and (5), by (18), (19), and (20), we have (6), (7), and (8). The theorem is proved.

Proof of Theorem 2. By Cauchy's inequality and (18), we have

$$\begin{aligned}
\left(\int_a^b \frac{1}{x+y} f(x)\,dx\right)^2 &= \left[\int_a^b \frac{1}{(x+y)^{1/2}}\left(\frac{x}{y}\right)^{1/4} f(x) \frac{1}{(x+y)^{1/2}}\left(\frac{y}{x}\right)^{1/4} dx\right]^2 \\
&\le \int_a^b \frac{1}{x+y}\left(\frac{x}{y}\right)^{1/2} f^2(x)\,dx \int_a^b \frac{1}{x+y}\left(\frac{y}{x}\right)^{1/2} dx \\
&= \omega_1(a,b,y)\int_a^b \frac{1}{x+y}\left(\frac{x}{y}\right)^{1/2} f^2(x)\,dx \\
&\le \pi\left(1-\sqrt[4]{\frac{a}{b}}\right)\int_a^b \frac{1}{x+y}\left(\frac{x}{y}\right)^{1/2} f^2(x)\,dx.
\end{aligned}$$

Then,

$$\begin{aligned}
\int_a^b\left(\int_a^b \frac{f(x)}{x+y}\,dx\right)^2 dy &\le \pi\left(1-\sqrt[4]{\frac{a}{b}}\right)\int_a^b\int_a^b \frac{1}{x+y}\left(\frac{x}{y}\right)^{1/2} f^2(x)\,dx\,dy \\
&= \pi\left(1-\sqrt[4]{\frac{a}{b}}\right)\int_a^b \omega_1(a,b,x)f^2(x)\,dx \\
&\le \pi^2\left(1-\sqrt[4]{\frac{a}{b}}\right)^2\int_a^b f^2(x)\,dx.
\end{aligned}$$

Hence (9) holds. Using (19) and (20), we can show that (10) and (11) are valid by the same method. The theorem is proved.

Remark 2. Still by (21) and (22), we have the further result as

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^t}\,dx\,dy \le \left[k_t - \eta_t(a,b)\right] \times \left(\int_a^b x^{1-t} f^2(x)\,dx \int_a^b x^{1-t} g^2(x)\,dx \right)^{1/2}, \quad (23)$$

where,

$$\eta_t(\alpha, b) = \min_{a \le x \le b} \left\{ \int_0^{a/x} \frac{1}{(1+u)^t} \left(\frac{1}{u}\right)^{1-t/2} du + \int_0^{x/b} \frac{1}{(1+u)^t} \left(\frac{1}{u}\right)^{1-t/2} du] \right\} > 2\theta_t \left(\frac{a}{b}\right)^{t/4} = k_t \left(\frac{a}{b}\right)^{t/4}.$$

It follows that $k_t[1 - (a/b)^{t/4}]$ in (3) is not the best possible and thus so are the constants in (6) and (9).

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